

A Switched Dynamical System Framework for Analysis of Massively Parallel Asynchronous Numerical Algorithms

Kooktae Lee, Raktim Bhattacharya, and Vijay Gupta

Abstract—In the near future, massively parallel computing systems will be necessary to solve computation intensive applications. The key bottleneck in massively parallel implementation of numerical algorithms is the synchronization of data across processing elements (PEs) after each iteration, which results in significant idle time. Thus, there is a trend towards relaxing the synchronization and adopting an asynchronous model of computation to reduce idle time. However, it is not clear what is the effect of this relaxation on the stability and accuracy of the numerical algorithm.

In this paper we present a new framework to analyze such algorithms. We treat the computation in each PE as a dynamical system and model the asynchrony as stochastic switching. The overall system is then analyzed as a switched dynamical system. However, modeling of massively parallel numerical algorithms as switched dynamical systems results in a very large number of modes, which makes current analysis tools available for such systems computationally intractable. We develop new techniques that circumvent this scalability issue. The framework is presented on a one-dimensional heat equation and the proposed analysis framework is verified by solving the partial differential equation (PDE) in a nVIDIA Tesla™ GPU machine, with asynchronous communication between cores.

I. INTRODUCTION

Exascale computing systems will soon be available to study computation intensive applications such as multi-physics multi-scale simulations of natural and engineering systems. Many scientific and practical problems can be described very accurately by ordinary or partial differential equations which may be tightly coupled with long-range correlations. These exascale systems may have $O(10^5 - 10^6)$ processors ranging from multicore processors to symmetric multiprocessors [1]–[3]. Furthermore, such systems are likely to be heterogeneous using both heavily multi-threaded CPUs as well as GPUs. Many challenges must be overcome before exascale systems can be utilized effectively in such applications. One such obstacle is the communication in tightly coupled problems during parallel implementation of any iterative numerical algorithm. This communication requires massive data movement in turn leading to idle time as the cores need to be synchronized after each time step.

Recent literature has proposed relaxing these synchronization requirements across the PEs [4]. This potentially eliminates the overhead associated with extreme parallelism

and significantly reduces computational time. However, the price to pay is loss of predictability possibly resulting in calculation errors. Thus, a rigorous analysis of the tradeoff between speed and accuracy is critical. This paper presents a framework for quantifying this tradeoff by analyzing the asynchronous numerical algorithm as a switched dynamical system [5]–[14]. While there is a large literature for analysis of such systems, these techniques are not applicable to our application. The reason is that due to the large number of PEs, the switched system model has an extremely large number of modes, which makes the available analysis tools intractable. Key contributions in this paper include new techniques for a) *stability analysis*, or quantification of steady-state error with respect to the synchronous solution; b) *convergence rate analysis* of the expected value of this error; and c) *probabilistic bounds* on this error. These techniques are developed to be computationally efficient, and avoid the aforementioned scalability issue.

The paper is organized as follows. Section II addresses the problems for the asynchronous numerical algorithm. In section III, we introduce a switched system framework to model the system structure for the asynchronous numerical scheme. The stability results are presented in section IV, and section V shows the convergence rate analysis. Then, the error analysis in probability is developed in section VI. Section VII demonstrates the usefulness of the proposed method by examples. Finally, section VIII concludes this paper.

II. PROBLEM FORMULATION

Notation: The symbol $\|\cdot\|$ and $\|\cdot\|_\infty$ stand for the Euclidean and infinity norm, respectively. The set of positive integers are denoted by \mathbb{N} . Further, $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$. Also, $\lambda(\cdot)$ represents an eigenvalue of a square matrix. In particular, $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the largest and the smallest eigenvalue in magnitude, respectively. The symbols \otimes , $\det(\cdot)$, $\text{tr}(\cdot)$, and $\text{vec}(\cdot)$ denote Kronecker product, matrix determinant, trace operator, and vectorization operator, respectively. Finally, the symbol $\Pr(\cdot)$ stands for the probability.

In this paper we demonstrate our framework and techniques on the one-dimensional heat equation, given by

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad (1)$$

where u is the time and space-varying state of the temperature, and t and x are continuous time and space respectively. The constant $\alpha > 0$ is the thermal diffusivity of the given material.

Kooktae Lee and Raktim Bhattacharya are with the Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843-3141, USA, {animodor, raktim}@tamu.edu. Vijay Gupta is with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA, vgupta2@nd.edu. Kooktae Lee and Raktim Bhattacharya were supported by NSF award #1349100, and Vijay Gupta was supported by NSF award #0846631.

The PDE is solved numerically using the finite difference method by Euler explicit scheme, with a forward difference in time and a central difference in space. Thus (1) is approximated as

$$\frac{u_i(k+1) - u_i(k)}{\Delta t} = \alpha \left(\frac{u_{i+1}(k) - 2u_i(k) + u_{i-1}(k)}{\Delta x^2} \right), \quad (2)$$

where $k \in \mathbb{N}_0$ is the discrete-time index and u_i is the temperature value at i^{th} grid space point. The symbols Δt and Δx denote the sampling time and the grid resolution in space, respectively. Further, if we define a constant $r \triangleq \alpha \frac{\Delta t}{\Delta x^2}$, then (2) can be written as

$$u_i(k+1) = ru_{i+1}(k) + (1-2r)u_i(k) + ru_{i-1}(k), \quad (3)$$

It is important to observe that (3) is a discrete-time linear dynamical system.

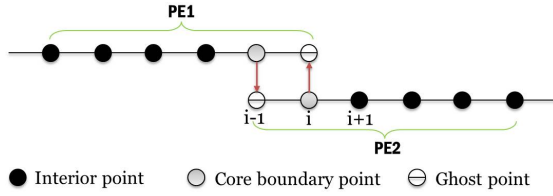


Fig. 1. Discretized one-dimensional domain with an asynchronous numerical algorithm. the PE denotes a group of grid points, assigned to each core.

Fig. 1 illustrates the numerical scheme over the discretized 1D spatial domain. A typical *synchronous* parallel implementation of this numerical scheme assigns several of these grid points to each PE. The updates for the temperature at the grid points assigned to each PE, occur in parallel. However, at every time step k , the data associated with the boundary grid points, where the communication is necessary are synchronized, and used to compute $u_i(k+1)$. This synchronization across PEs is slow, especially for massively parallel systems (estimates of idle time due to this synchronization give figures of up to 80% of the total time taken for the simulation as idle time). Recently, an alternative implementation which is *asynchronous* has been proposed. In this implementation, the updates in a PE occur without waiting for the other PEs to finish and their results to be synchronized. The data update across PEs occurs sporadically and independently. This asynchrony directly affects the update equation for the boundary points, as they depend on the grid points across PEs. For these points, the update is performed with the most recent available value, typically stored in a buffer. The effect of this asynchrony then propagates to other grid points. Within a PE, we assume there is no asynchrony and data is available in a common memory.

Thus, the asynchronous numerical scheme corresponding to (3) is given by

$$u_i(k+1) = ru_{i+1}(k_{i+1}^*) + (1-2r)u_i(k) + ru_{i-1}(k_{i-1}^*), \quad (4)$$

where $k_i^* \in \{k, k-1, k-2, \dots, k-q+1\}$, $i = 1, 2, \dots, N$, denotes the randomness caused by communication delays between PEs. The subscript i in k_i^* depicts that each grid space point may have different time delays. The parameter q is the length of a buffer that every core maintains to store data transmitted from the other cores. In this paper, we treat k_i^* as a random variable and thus (4) can be considered to be a linear discrete-time dynamical system with stochastic updates.

Although (4) is derived for the 1D heat equation, the treatment above can be developed for any parabolic PDEs. This observation encourages us to consider using tools from dynamical systems to analyze the effect of asynchrony in parallel numerical algorithms. Therefore, the primary goal of this study is to investigate the *stability*, *convergence rate*, and *error probability* of the asynchronous numerical algorithm in the framework of stochastic switched dynamical systems.

III. A SWITCHED SYSTEM APPROACH

Let us define the state vector $U_j(k) \in \mathbb{R}^n \triangleq [u_1^j(k), u_2^j(k), \dots, u_n^j(k)]^\top$, where $u_i^j(k)$ stands for the i^{th} grid space point in the j^{th} PE and n is the total number of grid points in the j^{th} PE. Therefore, (3) can be compactly written as

$$U(k+1) = AU(k), \quad k \in \mathbb{N}_0,$$

where $U(k) \in \mathbb{R}^{Nn} \triangleq [U_1(k)^\top, U_2(k)^\top, \dots, U_N(k)^\top]^\top$, N is the total number of PEs, n is the size of the state for each PE, and system matrix $A \in \mathbb{R}^{Nn \times Nn}$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ r & 1-2r & r & 0 & \dots & 0 \\ 0 & r & 1-2r & r & \dots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \dots & & r & 1-2r & r \\ 0 & \dots & & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{Nn \times Nn}.$$

Note that the first and the last row of A matrix specify the Dirichlet boundary conditions (see pp. 150, [15]). i.e., we have the constant in time boundary temperatures for simplicity.

Next, we define the augmented state $X(k) \in \mathbb{R}^{Nnq} \triangleq [U(k)^\top, U(k-1)^\top, \dots, U(k-q+1)^\top]^\top$, where, as stated before, q is the buffer length. For pedagogical simplicity (and without loss of generality), we consider the case with $q = 2$ and $N = 3$. Further, we let $n = 1$, which implies there is only one grid point in each PE. For this particular case, we construct the following matrices,

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 1-2r & r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2r & r & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \mathbf{I} & \mathbf{0} \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 1-2r & 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad W_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2r & 0 & r & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \mathbf{I} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{I} \in \mathbb{R}^{Nn \times Nn}$ and $\mathbf{0} \in \mathbb{R}^{Nn \times Nn}$ are the identity and the zero matrices with appropriate dimensions. As in [4], we assume that the condition $0 < r \leq 0.5$ holds from now on. The asynchronous numerical scheme can then be written as a switched system

$$X(k+1) = W_{\sigma_k} X(k), \quad \sigma_k \in \{1, 2, \dots, m\}, \quad k \in \mathbb{N}_0, \quad (5)$$

where the matrices $W_{\sigma_k} \in \mathbb{R}^{Nnq \times Nnq}$, are the subsystem dynamics. In general, the total number of switching modes is $m = q^{2(N-2)}$ that is obtained by considering all cases to distribute every components r in W_1 matrix, where the number of r in W_1 is given by $2(N-2)$, into q numbers of zero block matrix as in the above example. Therefore, the number of modes increase exponentially with the number of PEs, which is quite large for massively parallel systems.

At every time step, the numerical scheme evolves using one of the m modes, which depends on the variable k_i^* . In this paper, we model the variable k_i^* as a random variable that evolves in an independently and identically distributed (i.i.d.) fashion in time, and independently from one core to the next. Hence, we let π_j be the modal probability for W_j which is assumed to be stationary in time. Let $\Pi \triangleq \{\pi_1, \pi_2, \dots, \pi_m\}$, be the switching probabilities such that $0 \leq \pi_j \leq 1$, $\forall j$ and $\sum_{j=1}^m \pi_j = 1$. The system in (5) is thus an i.i.d jump linear system, which is a simpler case of the more well-known Markovian jump linear systems [13]. Even though the analysis theory for such systems is well developed, the existing tools are not suitable for our application because of the extremely large number of modes, particularly when N is large. Thus, we now develop an analysis theory for the i.i.d. jump linear systems which scales better with respect to the number of modes.

IV. STABILITY

The first requirement is that of convergence of (5). Because of the Dirichlet boundary conditions, we expect the temperature to converge to a constant value for every grid point. We proceed to analyze the conditions for convergence (or stability) of the system. To this end, we may try to use the infinity norm and apply the sub-multiplicative property to obtain $\|X(k+1)\|_\infty = \|W_{\sigma_k} X(k)\|_\infty \leq \|W_{\sigma_k}\|_\infty \|X(k)\|_\infty = \|X(k)\|_\infty$, where the last equality holds since we have $\|W_j\|_\infty = 1$, $\forall j$. This can be written as

$$\frac{\|X(k+1)\|_\infty}{\|X(k)\|_\infty} \leq 1. \quad (6)$$

The above result only shows that the solution from the asynchronous algorithm is *marginally* stable and we are unable to determine the steady-state solution.

In fact, we can show that the asynchronous scheme also attains the same steady-state value as the synchronous scheme, regardless of the specific realization of $\{\sigma_k\}$. Using spectral decomposition, the matrices W_j can be expressed in terms of the eigenvalues and corresponding eigenvectors as

$$W_j \in \mathbb{R}^{Nnq \times Nnq} = \sum_{i=1}^{Nnq} \lambda_i^j v_i^j s_i^j, \quad j = \{1, 2, \dots, m\}, \quad (7)$$

where $\lambda_i^j \in \mathbb{R}$, $v_i^j \in \mathbb{R}^{Nnq \times 1}$, and $s_i^j \in \mathbb{R}^{1 \times Nnq}$ denote the eigenvalues, right eigenvectors, and left eigenvectors of W_j , respectively.

Since $\max_i |\lambda_i^j| \leq \|W_j\|_\infty = 1$, $\forall j$, the spectral radius of W_j , $j = 1, 2, \dots, m$, is less than or equal to 1. Therefore, we may order the eigenvalues as $1 \geq |\lambda_1^j| \geq |\lambda_2^j| > \dots \geq |\lambda_{Nnq}^j| \geq 0$. It can be shown that all W_j have two eigenvalues with value 1, irrespective of the size of q and N . Therefore, the eigenvalues for W_j are ordered as $1 = |\lambda_1^j| = |\lambda_2^j| > |\lambda_3^j| \geq \dots \geq |\lambda_{Nnq}^j| \geq 0$.

Moreover, the left and right eigenvectors for eigenvalues equal to 1 are common eigenvectors for all matrices W_j , $j = 1, 2, \dots, m$. These common left and right eigenvectors are

1) Left eigenvectors:

$$s_1 = [1, 0, \dots, 0, \mathbf{0}, \dots, \mathbf{0}] \in \mathbb{R}^{1 \times Nnq}, \quad (8)$$

$$s_2 = [0, \dots, 0, 1, \mathbf{0}, \dots, \mathbf{0}] \in \mathbb{R}^{1 \times Nnq}, \quad (9)$$

2) Right eigenvectors:

$$v_1 = [\mu_1, \mu_1, \dots, \mu_1]^\top \in \mathbb{R}^{Nnq \times 1}, \quad (10)$$

$$v_2 = [\mu_2, \mu_2, \dots, \mu_2]^\top \in \mathbb{R}^{Nnq \times 1}, \quad (11)$$

where $\mathbf{0} \in \mathbb{R}^{1 \times Nn}$ denotes a row vector with all zero elements, and $\mu_1 \triangleq [1, \frac{Nn-2}{Nn-1}, \dots, \frac{Nn-j}{Nn-1}, \dots, \frac{1}{Nn-1}, 0] \in \mathbb{R}^{1 \times Nn}$, $\mu_2 \triangleq [0, \frac{1}{Nn-1}, \frac{2}{Nn-1}, \dots, \frac{j-1}{Nn-1}, \dots, \frac{Nn-2}{Nn-1}, 1] \in \mathbb{R}^{1 \times Nn}$, $j = 1, 2, \dots, Nn$.

Notice that we have $W_j v_i = v_i$ and $s_i W_j = s_i$, $i = 1, 2$, $\forall j$. Then, the steady-state value for the asynchronous scheme is given by the following result.

Proposition 4.1: Consider the i.i.d. jump linear system in (5) with subsystem matrices W_j , $j = 1, 2, \dots, m$ and a stationary switching probability Π . For a given initial condition $X(0)$, if we define $\Psi \triangleq v_1 s_1 + v_2 s_2$, where v_i and s_i , $i = 1, 2$, are given in (8)–(11), then, the steady-state value X_{ss} has the following form:

$$X_{ss} \triangleq \lim_{k \rightarrow \infty} X(k) = \Psi X(0),$$

irrespective of the switching sequence $\{\sigma_k\}$.

Proof: Let the eigenvalues of W_j be ordered in magnitude by $1 = |\lambda_1^j| = |\lambda_2^j| > |\lambda_3^j| \geq \dots \geq |\lambda_{Nnq}^j| \geq 0$. Also, let v_i^j and s_i^j be the right and left eigenvector corresponding to λ_i^j , respectively. Using the spectral decomposition, W_j can be alternatively expressed by $W_j = \sum_{i=1}^{Nnq} \lambda_i^j v_i^j s_i^j = \Psi + \sum_{\lambda_i^j \neq 1} f^j(i)$, where $\Psi \triangleq v_1 s_1 + v_2 s_2$ and $f^j(i) \triangleq \lambda_i^j v_i^j s_i^j$.

Then, starting with $X(0)$, the realization of the switching sequence σ_k results in

$$\begin{aligned} X(k) &= W_{\sigma_{k-1}} W_{\sigma_{k-2}} \dots W_{\sigma_1} W_{\sigma_0} X(0) \\ &= \left(\Psi + \sum_{\lambda_i^{\sigma_{k-1}} \neq 1} f^{\sigma_{k-1}}(i) \right) \dots \left(\Psi + \sum_{\lambda_i^{\sigma_0} \neq 1} f^{\sigma_0}(i) \right) X(0) \\ &= \left(\Psi^k + g(k) \right) X(0), \end{aligned}$$

where in above equation, $g(k)$ represents all the other multiplication terms except Ψ^k term. Note that $g(k)$ is formed

by the product of λ_i^j , where $0 \leq |\lambda_i^j| < 1$, $\forall i > 2$, $\forall j$. Consequently, if $k \rightarrow \infty$, then $g(k)$ is asymptotically convergent to zero since the *infinite* number of multiplication of the term λ_i^j , $\forall i > 2$, converges to zero. Therefore, we have

$$X_{ss} = \lim_{k \rightarrow \infty} X(k) = \lim_{k \rightarrow \infty} \Psi^k X(0) = \Psi X(0).$$

The last equality in above equation holds because $\Psi^k = \Psi^{k-1} = \dots = \Psi$, $\forall k \in \mathbb{N}$. ■

V. CONVERGENCE RATE

In this section, we investigate how fast the expected value of the state converges to the steady-state X_{ss} by analyzing the transient behavior of the asynchronous algorithm. Let us define a new state variable $e(k) \triangleq X(k) - X_{ss}$. The expected value of $e(k)$ is given by $\bar{e}(k) \triangleq \mathbb{E}[X(k) - X_{ss}] = \mathbb{E}[X(k)] - X_{ss} = \bar{X}(k) - X_{ss}$, where $\bar{X}(k) \triangleq \mathbb{E}[X(k)]$. Therefore, the convergence rate of $\|\bar{e}(k)\|$ will provide bound for the convergence rate of $\|\bar{X}(k) - X_{ss}\|$.

To obtain an upper bound for the convergence rate of $\|\bar{e}(k)\|$, we use the following matrix transformation. As described in (7), each modal matrix W_j can be alternatively expressed by $W_j = \sum_{i=1}^{Nnq} \lambda_i^j v_i^j s_i^j$, where λ_i^j , v_i^j , and s_i^j denote the eigenvalues, right and, respectively, left eigenvectors for W_j . If we define the transformed matrix $\tilde{W}_j \triangleq W_j - \sum_{\lambda_i^1=1} \lambda_i^j v_i^j s_i^j = W_j - \Psi = \sum_{\lambda_i^1 \neq 1} \lambda_i^j v_i^j s_i^j$, then the modal dynamics with the corresponding state $e_j(k)$, is given by

$$e_j(k+1) = \tilde{W}_j e_j(k), \quad j = \{1, 2, \dots, m\}, \quad k \in \mathbb{N}_0. \quad (12)$$

Moreover, as in (5), the error state $e(k) = X(k) - X_{ss}$, is governed by

$$e(k+1) = \tilde{W}_{\sigma_k} e(k), \quad \sigma_k \in \{1, 2, \dots, m\}, \quad k \in \mathbb{N}_0. \quad (13)$$

The system in (13) is also a switched linear system. The transformed matrix \tilde{W}_j are the modes of the error dynamics. Generally, it is difficult to estimate the convergence rate of the ensemble with stochastic jumps. Previous works [16]–[19] have used the common Lyapunov function approaches, to analyze stability and the convergence rate. However, the existence of a common Lyapunov function is the *only sufficient condition* for the system stability, and hence there may not exist a common Lyapunov function for the asynchronous algorithm. Moreover, extremely large values of m make it very difficult to test every conditions for the existence of such a common Lyapunov function. For this reason, we bound the convergence rate of $\bar{e}(k)$, instead of bounding $e(k)$ directly.

Lemma 5.1: Consider an i.i.d. jump linear system given by (13) with the switching probability $\Pi = \{\pi_1, \pi_2, \dots, \pi_m\}$. If the initial state $e(0)$ is given and has no uncertainty, the expected value of $e(k)$ is updated by

$$\bar{e}(k) \triangleq \mathbb{E}[e(k)] = \Lambda^k e(0) \quad \text{or} \quad \bar{e}(k+1) = \Lambda \bar{e}(k), \quad (14)$$

where $\Lambda \triangleq \sum_{i=1}^m \pi_i \tilde{W}_i$.

Proof: For an i.i.d. jump process with a given deterministic initial error $e(0)$, we have

$$\begin{aligned} \mathbb{E}[e(k)] &= \mathbb{E}[\tilde{W}_{\sigma_{k-1}} e(k-1)] \\ &= \mathbb{E}[\tilde{W}_{\sigma_{k-1}} \tilde{W}_{\sigma_{k-2}} \dots \tilde{W}_{\sigma_1} \tilde{W}_{\sigma_0} e(0)] \\ &= \underbrace{\mathbb{E}[\tilde{W}_{\sigma_{k-1}}]}_{=\Lambda} \dots \underbrace{\mathbb{E}[\tilde{W}_{\sigma_1}]}_{=\Lambda} \underbrace{\mathbb{E}[\tilde{W}_{\sigma_0}]}_{=\Lambda} e(0) = \Lambda^k e(0). \end{aligned}$$

Since the matrix Λ is given by $\Lambda = \sum_{i=1}^m \pi_i \tilde{W}_i$, the computation of Λ requires all matrices W_j , $j = 1, 2, \dots, m$. As pointed out earlier, this calculation is intractable due to the extremely large number of the switching modes m . Therefore, instead of using (14), we provide a computationally efficient method to bound $\|\bar{e}(k)\|$ through a Lyapunov theorem.

Consider a discrete-time Lyapunov function $V(k) = \bar{e}(k)^\top P \bar{e}(k)$, where P is a positive definite matrix. Since it is shown that the original state $X(k)$ is convergent to the unique steady-state X_{ss} as $k \rightarrow \infty$ irrespective of $\{\sigma_k\}$, the expected error $\bar{e}(k) \triangleq \bar{X}(k) - X_{ss}$ is asymptotically stable. Therefore, one can employ the *Converse Lyapunov Theorem* [20], which guarantees the existence of a positive definite matrix P , satisfying the following linear matrix inequality (LMI) condition $\Lambda^\top P \Lambda - P < -Q$, where Q is some positive definite matrix. The matrix inequality can be interpreted in the sense of positive definiteness. (i.e., $A > B$ means the matrix $A - B$ is positive definite.) Then, the above LMI condition results in $\Delta V(k) = V(k+1) - V(k) = \bar{e}(k)^\top (\Lambda^\top P \Lambda - P) \bar{e}(k) < -\bar{e}(k)^\top Q \bar{e}(k) \leq -\lambda_{\min}(Q) \|\bar{e}(k)\|^2$. Also, the Lyapunov function $V(k)$ satisfies

$$\lambda_{\min}(P) \|\bar{e}(k)\|^2 \leq V(k) \leq \lambda_{\max}(P) \|\bar{e}(k)\|^2,$$

resulting in $-\|\bar{e}(k)\|^2 \leq -\frac{V(k)}{\lambda_{\max}(P)}$. Therefore, we have

$$\begin{aligned} \Delta V(k) &< -\lambda_{\min}(Q) \|\bar{e}(k)\|^2 \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(k). \\ \Rightarrow V(k+1) &< \left(1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\right) V(k). \end{aligned} \quad (15)$$

Hence, $\|\bar{e}(k)\|$ is bounded by a following equation:

$$\|\bar{e}(k)\|^2 < K \left(1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\right)^k \|e(0)\|^2, \quad (16)$$

where $K > 0$ is some constant.

Next, we bound the convergence rate for $\|\bar{e}(k)\|$ by using the result in (16) as follows.

Proposition 5.1: For a *stable* i.i.d. jump linear system (13) with a stationary switching probability Π , consider a Lyapunov candidate function for the state \bar{e} , given by $V \triangleq \bar{e}^\top P \bar{e}$, where P is a positive definite matrix. In addition, a Lyapunov candidate function for (12) is given by $V_j \triangleq e_j^\top P_j e_j$, $j = 1, 2, \dots, m$, where P_j is a positive definite matrix. According to the Converse Lyapunov Theorem, there exist $P_j > 0$ and $P > 0$ such that $\tilde{W}_j^\top P_j \tilde{W}_j - P_j < -Q_j$, $j = 1, 2, \dots, m$ and $\Lambda^\top P \Lambda - P < -Q$, where Q_j and

Q are any positive definite matrices. Then, with a particular choice of these matrices, we assume that P_j and P satisfy the following conditions:

$$\tilde{W}_j^\top P_j \tilde{W}_j - P_j = -I, \quad j = 1, 2, \dots, m, \quad (17)$$

$$\Lambda^\top P \Lambda - P \leq -\varepsilon_j I, \quad \text{for some } j, \quad (18)$$

where $\varepsilon_j \triangleq \frac{\lambda_{\max}(P)}{\lambda_{\max}(P_j)} > 0$, \tilde{W}_j are the modal matrices in (12), and $\Lambda \triangleq \sum_{j=1}^m \pi_j \tilde{W}_j$.

Then, $\|\bar{e}(k)\|^2$ is bounded by

$$\|\bar{e}(k)\|^2 < K \left(1 - \frac{1}{\lambda_{\max}(P_j)}\right)^k \|e(0)\|^2, \quad (19)$$

where $K > 0$ is some constant.

Proof: By applying the result in (16) into (18), we have

$$\begin{aligned} \|\bar{e}(k)\|^2 &< K \left(1 - \frac{\lambda_{\min}(\varepsilon_j I)}{\lambda_{\max}(P)}\right)^k \|e(0)\|^2 \\ &= K \left(1 - \frac{\varepsilon_j}{\lambda_{\max}(P)}\right)^k \|e(0)\|^2 \\ &= K \left(1 - \frac{1}{\lambda_{\max}(P_j)}\right)^k \|e(0)\|^2. \end{aligned}$$

The last equality in above equation holds by the definition of ε_j . ■

Proposition 5.1 says that we can always guarantee the bound for $\|\bar{e}(k)\|$ if (18) holds. Consequently, the existence of such a P , satisfying (18) is the major concern in order to guarantee the bound $\|\bar{e}(k)\|$. The following lemma and theorem can be used to prove the existence of such a P .

Lemma 5.2: Suppose that P_j is a positive definite matrix, satisfying (17). Then, the largest eigenvalue of P_j is strictly greater than 1 for all j , i.e., $\lambda_{\max}(P_j) > 1$, $\forall j$.

Proof: From (17), $P_j = \tilde{W}_j^\top P_j \tilde{W}_j + I$, $\forall j$. Then, with the eigenvectors $y \in \mathbb{R}^{N_{nq}}$ of P_j , the largest eigenvalue of P_j is given by its definition as follows:

$$\begin{aligned} \lambda_{\max}(P_j) &= \lambda_{\max}(\tilde{W}_j^\top P_j \tilde{W}_j + I) \\ &= \max_{\|y\|^2=1} y^\top (\tilde{W}_j^\top P_j \tilde{W}_j + I) y \\ &= \max_{\|y\|^2=1} \left(y^\top \tilde{W}_j^\top P_j \tilde{W}_j y \right) + \underbrace{y^\top y}_{=\|y\|^2=1} \\ &= \max_{\|y\|^2=1} \left(y^\top \tilde{W}_j^\top P_j \tilde{W}_j y \right) + 1 \end{aligned}$$

Since P_j is a positive definite matrix, $\tilde{W}_j^\top P_j \tilde{W}_j$ becomes a positive semi-definite matrix at least. Then, the scalar term $y^\top \tilde{W}_j^\top P_j \tilde{W}_j y$ cannot be zero unless $\tilde{W}_j^\top P_j \tilde{W}_j$ is a zero matrix or a triangular matrix with zero diagonal components, which is not the case. Hence, it is guaranteed that $y^\top \tilde{W}_j^\top P_j \tilde{W}_j y > 0$, implying $\lambda_{\max}(P_j) > 1$, $\forall j$. ■

Theorem 5.1: Consider Lyapunov functions for (12) and (13) given by $V_j \triangleq e_j^\top P_j e_j$, $j = 1, 2, \dots, m$, and $V \triangleq \bar{e}^\top P \bar{e}$, respectively, where the matrices $P_j > 0$, $\forall j$ and $P > 0$. By the Converse Lyapunov Theorem, we assume that the matrices P_j , $\forall j$, satisfies the condition (17).

Then, there exists a positive definite matrix P such that

$$\Lambda^\top P \Lambda - P \leq -\varepsilon_j I, \quad \text{for some } j, \quad (20)$$

where $\varepsilon_j \triangleq \frac{\lambda_{\max}(P)}{\lambda_{\max}(P_j)} > 0$.

Proof: We prove by contradiction. Suppose that there exist no such $P > 0$, satisfying (20), which is equivalent to that for **all** matrices $P > 0$, the inequality $\Lambda^\top P \Lambda - P > -\varepsilon_j I$ holds $\forall j$. The above inequality can be interpreted in the quadratic sense. In other words, for any non-zero vector v that has a proper dimension, the following condition holds:

$$v^\top (\Lambda^\top P \Lambda - P + \varepsilon_j I) v > 0, \quad \forall j \quad (21)$$

As a particular choice of v , we let the vector v be the eigenvector of the matrix Λ , i.e., $\Lambda v = \lambda v$, where λ is the eigenvalue of Λ . Since (21) holds for any matrix $P > 0$, we let $P = I$, which results in $\varepsilon_j = \frac{\lambda_{\max}(I)}{\lambda_{\max}(P_j)} = \frac{1}{\lambda_{\max}(P_j)}$. Hence, we have

$$\begin{aligned} 0 &< v^\top \left(\Lambda^\top \Lambda - I + \frac{1}{\lambda_{\max}(P_j)} I \right) v \\ &= \underbrace{(\Lambda v)^\top}_{=\lambda v} \underbrace{(\Lambda v)}_{=\lambda v} - \|v\|^2 + \frac{1}{\lambda_{\max}(P_j)} \|v\|^2 \\ &= \left(\lambda^2 - 1 + \frac{1}{\lambda_{\max}(P_j)} \right) \|v\|^2, \quad \forall j. \end{aligned}$$

From the structure of the matrix Λ , it can be shown that $\det(\Lambda) = 0$. Therefore, one of the eigenvalues λ is zero. Moreover, Lemma 5.2 states that $\frac{1}{\lambda_{\max}(P_j)} < 1$, $\forall j$. As a consequence, with $\lambda = 0$, we have

$$0 < \underbrace{\left(-1 + \frac{1}{\lambda_{\max}(P_j)} \right)}_{<0} \underbrace{\|v\|^2}_{>0} < 0, \quad \forall j.$$

which is a *contradiction*. ■

Remark 5.1: Proposition 5.1 provides a very efficient way to bound the convergence rate for $\|\bar{e}(k)\|$. According to the proposed methods, it is unnecessary to compute the matrix Λ and to keep all matrices W_j , $j = 1, 2, \dots, m$ since $\|\bar{e}(k)\|$ is bounded by the proposed Lyapunov function. Also, Theorem 5.1 guarantees the condition (18), which is assumed in Proposition 5.1.

Note that we specify the modal matrix W_m in (5) as the most delayed case – all PEs use the oldest value in the buffer. Therefore, it can be inferred that $\lambda_{\max}(P_m) \geq \lambda_{\max}(P_j)$, $\forall j$, which results in

$$\|\bar{e}(k)\|^2 < K \left(1 - \frac{1}{\lambda_{\max}(P_m)}\right)^k \|e(0)\|^2, \quad (22)$$

where K is a positive constant. Therefore, the *only information* required to compute the convergence rate of $\|\bar{e}(k)\|$, is the matrix W_m with the corresponding positive definite matrix P_m . As a result, the rate of convergence can be calculated by the proposed methods *without any scalability problems*.

VI. ERROR ANALYSIS

In this section, we investigate the error probability, which quantifies the deviation of the random vector $X(k)$ from its steady-state value X_{ss} in probability. To measure this error probability, the Markov inequality given by $\Pr(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}$, where X is a nonnegative random variable and ϵ is a positive constant, is used. First of all, we investigate the term $\text{vec}(e(k)e(k)^\top)$ as follows:

$$\begin{aligned} \text{vec}(e(k)e(k)^\top) &= \text{vec}(\tilde{W}_{\sigma_{k-1}}e(k-1)e(k-1)^\top\tilde{W}_{\sigma_{k-1}}^\top) \\ &= (\tilde{W}_{\sigma_{k-1}} \otimes \tilde{W}_{\sigma_{k-1}})\text{vec}(e(k-1)e(k-1)^\top). \end{aligned} \quad (23)$$

In the second equality of above equation, we used the property that $\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B)$.

By taking the expectation with new definitions $y(k) \triangleq \text{vec}(e(k)e(k)^\top)$, $\bar{y}(k) \triangleq \mathbb{E}[y(k)]$, and $\Gamma_{\sigma_k} \triangleq \tilde{W}_{\sigma_k} \otimes \tilde{W}_{\sigma_k}$, (23) becomes

$$\begin{aligned} \bar{y}(k) &\triangleq \mathbb{E}[y(k)] = \mathbb{E}[\Gamma_{\sigma_{k-1}}y(k-1)] \\ &= \sum_{r=1}^m \mathbb{E}[\Gamma_{\sigma_{k-1}}y(k-1) \mid \sigma_{k-1} = r] \Pr(\sigma_{k-1} = r) \\ &= \sum_{r=1}^m \pi_r \Gamma_r \mathbb{E}[y(k-1)], \end{aligned}$$

resulting in $\bar{y}(k) = (\sum_{r=1}^m \pi_r \Gamma_r) \bar{y}(k-1)$, where in the second line we applied the law of total probability and the last equality holds by $\Pr(\sigma_{k-1} = r) = \pi_r$ for i.i.d. switching.

By the exactly same argument given in Lemma 5.1 and Proposition 5.1, the upper bound for $\bar{y}(k)$ is obtained as follows:

$$\|\bar{y}(k)\| < K \left(1 - \frac{1}{\lambda_{\max}(\tilde{P}_m)}\right)^{k/2} \|y(0)\|, \quad \forall k \in \mathbb{N}, \quad (24)$$

where K is some positive constant and \tilde{P}_m is a positive definite matrix, satisfying the condition $\Gamma_m^\top \tilde{P}_m \Gamma_m - \tilde{P}_m = -I$. However, unlike the positive definite matrix $P_m \in \mathbb{R}^{Nnq \times Nnq}$ in (17), the dimension of the matrix \tilde{P}_m is given by $\tilde{P}_m \in \mathbb{R}^{(Nnq)^2 \times (Nnq)^2}$, which may be large in size, and hence incurs computational intractabilities to obtain such a \tilde{P}_m . Therefore, we introduce the following proposition and theorem in order to further facilitate the computation of $\lambda_{\max}(\tilde{P}_m)$ as follows.

Proposition 6.1: Consider a positive definite matrix \tilde{P}_m , satisfying the condition $\Gamma_m^\top \tilde{P}_m \Gamma_m - \tilde{P}_m = -I$, where $\Gamma_m \triangleq \tilde{W}_m \otimes \tilde{W}_m$, and \tilde{W}_m is any real square matrix. If we assume that there exist finite, positive constants k_0 , c_0 , and c_1 such that

$$1 \leq \|\tilde{W}_m^k\|^4 \leq c_0, \quad \text{for } k \in [0, k_0), \quad (25)$$

$$\|\tilde{W}_m^k\|^4 \leq c_1 < 1, \quad \text{for } k \in [k_0, \infty), \quad (26)$$

then, the largest eigenvalue of \tilde{P}_m is bounded by the following function:

$$\lambda_{\max}(\tilde{P}_m) < \sum_{k=0}^{\infty} \|\tilde{W}_m^k\|^4 \leq k_0 c_0 \left(\frac{1}{1 - c_1} \right), \quad (27)$$

Proof: The leftmost inequality in (27) can be proved as follows. The positive definite matrix \tilde{P}_m satisfying the condition $\Gamma_m^\top \tilde{P}_m \Gamma_m - \tilde{P}_m = -I$, is analytically computed by $\tilde{P}_m = \sum_{k=0}^{\infty} \left(\Gamma_m^{\top k} \right) I \left(\Gamma_m^k \right) = \sum_{k=0}^{\infty} \Gamma_m^{\top k} \Gamma_m^k$. Then, for a given matrix $\Gamma_m \triangleq \tilde{W}_m \otimes \tilde{W}_m$, we have

$$\begin{aligned} \Gamma_m^{\top k} \Gamma_m^k &< \rho(\Gamma_m^{\top k} \Gamma_m^k) I = \rho(\Gamma_m^k)^\top \Gamma_m^k I = \sigma_{\max}^2(\Gamma_m^k) I \\ &= \|\Gamma_m^k\|^2 I = \|(\tilde{W}_m \otimes \tilde{W}_m)^k\|^2 I = \|\tilde{W}_m^k\|^4 I, \end{aligned} \quad (28)$$

where $\rho(\cdot)$ and $\sigma_{\max}(\cdot)$ denote the spectral radius and the spectral norm, respectively. For equality conditions in (28), we used the known property that $\sqrt{\rho(\Gamma_m^k)^\top \Gamma_m^k} = \sigma_{\max}(\Gamma_m^k) = \|\Gamma_m^k\|$ and $\|(\tilde{W}_m \otimes \tilde{W}_m)^k\| = \|\tilde{W}_m^k \otimes \tilde{W}_m^k\| = \|\tilde{W}_m^k\|^2$, $\forall k \in \mathbb{N}_0$. By summing up from $k = 0$ to ∞ , and then taking the largest eigenvalue in (28), we have $\lambda_{\max}(\tilde{P}_m) = \lambda_{\max} \left(\sum_{k=0}^{\infty} \Gamma_m^{\top k} \Gamma_m^k \right) < \sum_{k=0}^{\infty} \|\tilde{W}_m^k\|^4$.

For the rightmost inequality in (27), the assumptions in (25)-(26) result in

$$\begin{aligned} \sum_{k=0}^{\infty} \|\tilde{W}_m^k\|^4 &= \underbrace{\sum_{k=0}^{k_0-1} \|\tilde{W}_m^k\|^4}_{\leq k_0 c_0} + \sum_{k=k_0}^{\infty} \|\tilde{W}_m^k\|^4 \\ &\leq k_0 c_0 + \sum_{k=k_0}^{2k_0-1} \|\tilde{W}_m^k\|^4 + \sum_{k=2k_0}^{3k_0-1} \|\tilde{W}_m^k\|^4 + \dots \\ &= k_0 c_0 + \sum_{k=0}^{k_0-1} \|\tilde{W}_m^{(k_0+k)}\|^4 + \sum_{k=2k_0}^{3k_0-1} \|\tilde{W}_m^k\|^4 + \dots \\ &\leq k_0 c_0 + \underbrace{\|\tilde{W}_m^{k_0}\|^4}_{\leq c_1} \underbrace{\sum_{k=0}^{k_0-1} \|\tilde{W}_m^k\|^4}_{\leq k_0 c_0} + \sum_{k=0}^{k_0-1} \|\tilde{W}_m^{(2k_0+k)}\|^4 + \dots \\ &\leq k_0 c_0 + k_0 c_0 c_1 + \underbrace{\|\tilde{W}_m^{2k_0}\|^4}_{\leq c_1^2} \underbrace{\sum_{k=0}^{k_0-1} \|\tilde{W}_m^k\|^4}_{\leq k_0 c_0} + \dots \\ &\leq k_0 c_0 + k_0 c_0 c_1 + k_0 c_0 c_1^2 + \dots \\ &= k_0 c_0 \left(\sum_{n=0}^{\infty} c_1^n \right) = k_0 c_0 \left(\frac{1}{1 - c_1} \right). \end{aligned}$$

Hence, we have $\sum_{k=0}^{\infty} \|\tilde{W}_m^k\|^4 \leq k_0 c_0 \left(\frac{1}{1 - c_1} \right)$. \blacksquare

Theorem 6.1: Consider a stable, i.i.d. jump linear system with subsystem dynamics \tilde{W}_j given in (13). Then, the probability of $\|e(k)\|^2 > \epsilon$, where ϵ is some positive constant, is given by

$$\Pr \left(\|e(k)\|^2 > \epsilon \right) \leq \min(1, \beta), \quad k \in \mathbb{N}_0, \quad (29)$$

where $\beta \triangleq \frac{\sqrt{n}K}{\epsilon} \left(1 - \frac{1 - c_1}{k_0 c_0} \right)^{k/2} \|y(0)\|$, $K > 0$ is a constant, c_0, c_1, k_0 are positive constants such that the conditions (25)-(26) are satisfied.

Proof: At first, we consider the following equality condition given by

$$\begin{aligned} \|e(k)\|^2 &= e(k)^\top e(k) = \text{tr}(e(k)^\top e(k)) = \text{tr}(I(e(k)e(k)^\top)) \\ &= \text{vec}(I)^\top \text{vec}(e(k)e(k)^\top) = \text{vec}(I)^\top y(k), \end{aligned} \quad (30)$$

where we used the cyclic permutation property for the trace operator in the first line and the equality in the second line holds by the property $\text{tr}(X^\top Y) = \text{vec}(X)^\top \text{vec}(Y)$ for any square matrix $X, Y \in \mathbb{R}^{n \times n}$.

We take the expectation in both sides of (30), which leads to

$$\mathbb{E}[\|e(k)\|^2] = \text{vec}(I)^\top \mathbb{E}[y(k)] = \text{vec}(I)^\top \bar{y}(k). \quad (31)$$

Since the term $\mathbb{E}[\|e(k)\|^2]$ is a scalar value, taking the Euclidean norm returns the same value. Hence, applying the Euclidean norm in (31) results in

$$\begin{aligned} \mathbb{E}[\|e(k)\|^2] &= \|\text{vec}(I)^\top \bar{y}(k)\| \\ &\leq \|\text{vec}(I)^\top\| \cdot \|\bar{y}(k)\| = \sqrt{n} \cdot \|\bar{y}(k)\|. \end{aligned} \quad (32)$$

Now, plugging (24) and (27) into (32) leads to

$$\mathbb{E}[\|e(k)\|^2] < \sqrt{n}K \left(1 - \frac{1 - c_1}{k_0 c_0}\right)^{k/2} \|y(0)\|.$$

Finally, by applying the Markov inequality the above equation ends up with

$$\Pr\left(\|e(k)\|^2 > \epsilon\right) \leq \frac{\mathbb{E}[\|e(k)\|^2]}{\epsilon} < \beta,$$

where $\beta \triangleq \frac{\sqrt{n}K}{\epsilon} \left(1 - \frac{1 - c_1}{k_0 c_0}\right)^{k/2} \|y(0)\|$.

Since the probability cannot exceed one, we have

$$\Pr\left(\|e(k)\|^2 > \epsilon\right) \leq \min(1, \beta) \quad \blacksquare$$

Theorem 6.1 represents the error probability for a given bound ϵ . Since $e(k)$ is a time-varying variable, the probability $\Pr(\|e(k)\|^2 > \epsilon)$ also changes with respect to time. Starting from a given initial condition $y(0)$, this probability will converge to zero if $\left(1 - \frac{1 - c_1}{k_0 c_0}\right) < 1$.

VII. SIMULATIONS

In order to test the proposed methods, simulation was carried out for the one-dimensional heat equation. We implemented the asynchronous parallel algorithm with CUDA C++ programming on nVIDIA TeslaTM C2050 GPU, which has 448 CUDA cores. The simulations were performed with the following parameters:

- Simulation Parameters:

$$\Delta x = 0.1, \Delta t = 0.01, \alpha = 0.5, r = \alpha \frac{\Delta t}{\Delta x^2} = 0.5$$

$$I.C.: \quad u_i = \cos^2\left(\frac{3\pi i}{2(N-1)}\right), \quad i = 1, 2, \dots, N$$

$$B.C.: \quad u_1(k) = 1, u_N(k) = 0, \quad \forall k$$

- Buffer length: $q = 3$

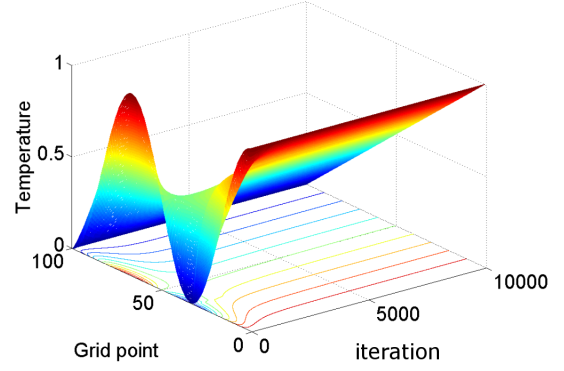


Fig. 2. The spatio-temporal change of the temperature. Initially, the temperature was given by the cosine square function. The total grid points are 100, and the simulation was terminated when $k = 10000$.

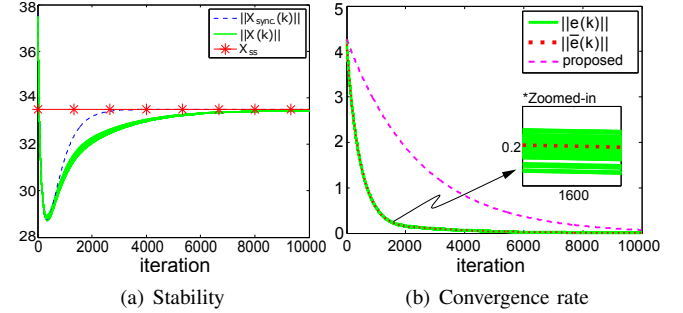


Fig. 3. The results for the stability and convergence rate. (a) The solid lines represent the ensembles of total 300 simulations. The synchronous case is given by dashed line. The steady-state is depicted by starred line. (b) The solid and dotted lines represent 300 ensembles for $\|e(k)\|$ and the normed empirical mean $\|\bar{e}(k)\|$, respectively. The dashed line shows the upper bound of $\|\bar{e}(k)\|$ from the proposed Lyapunov function, respectively.

- Number of PEs: $N = 100$.
- Number of grid points in PE: $n = 1$

For a given initial temperature, the spatio-temporal evolution of the state is presented in Fig. 2. As time k increases, the curved shape of the temperature, given as a cosine square function initially, flattens out. This simulation represents the synchronous case.

In Fig. 3 (a), the ensemble of the trajectories is shown for the asynchronous algorithm. The solid lines show the trajectories of total 300 simulations. Due to the randomness in the asynchronous algorithm, the trajectories differ from each other. For a reference, the synchronous scheme is also shown by a dashed line. Although it seems that the synchronous scheme converges faster with respect to the given iteration step, the physical simulation time may take more because the idle time is necessary at each iteration in the synchronous case. As the proposed method guarantees the stability through the common eigenvectors, both synchronous and asynchronous trajectories converged to the same steady-state value X_{ss} , depicted by starred line.

Next, we present the result for the convergence rate of the asynchronous algorithm. We assume that the switching probability Π has the form of an i.i.d. jump process. Fig. 3

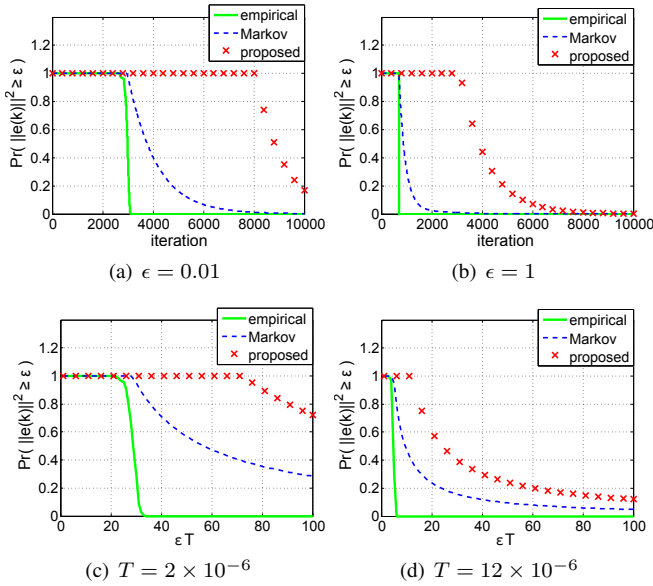


Fig. 4. Error probability with respect to time (a), (b) and with respect to ϵ (c), (d). The solid line and dashed line represent empirical error probability and empirical Markov inequality, respectively. The cross symbol denotes the upper bound for the error probability by the proposed method.

(b) shows the convergence rate of $\|\bar{e}(k)\|$, which describes how fast the expected value of the state converges to X_{ss} . The solid lines are 300 sample trajectories of $\|e(k)\|$, starting from the given initial condition: $e(0) = X(0) - X_{ss}$. The dotted line depicts the time history of the normed empirical mean $\|\bar{e}(k)\|$, whereas the dashed line shows an upper bound by the proposed Lyapunov method (22). Note that $\|e(k)\|$ is a random variable, and hence the normed empirical mean $\|\bar{e}(k)\|$ was obtained by averaging the data over 300 simulations. In the proposed method, however, it is not necessary to execute the simulation multiple times.

Fig. 4 represents the result for the error probability with respect to time and ϵ . For different values of ϵ , Fig. 4 (a) and (b) describe the time history of the error probabilities. The solid line denotes the empirical probability obtained from data – i.e., the number of samples satisfying $\|e(k)\|^2 > \epsilon$ divided by the total number of samples. The dashed line depicts the Markov inequality, computed from $\frac{\mathbb{E}[\|e(k)\|^2]}{\epsilon}$, where $\mathbb{E}[\|e(k)\|^2]$ is obtained by the statistics. Finally, the cross symbols mean the upper bound by the proposed method. As shown in Fig. 4 (a) and (b), the probabilities for all cases converge to zero since the error is asymptotically convergent.

On the other hands, Fig. 4 (c), (d) show the error probability with respect to ϵ at fixed time instance. In this result, the time is fixed at $k = 9000$ out of total 10000 iteration times, and the probability is computed while increasing ϵ values. In Fig. 4 (c) and (d), ϵT is given by the index along x -axis, where the value of T is given in Fig. 4 (c) and (d), respectively. In both cases, the error probabilities decrease as ϵ increases.

Although the proposed methods provide a conservative bound, it does not require executing the code multiple times

to predict the convergence rate or the error probability. In addition to that, the proposed methods are carried out in a computationally efficient manner without storing all subsystem matrices. In this example, we have $m = 3^{2(100-2)} \approx 3^{200}$, and keeping 3^{200} numbers of matrices is intractable in the real implementation. The proposed method, however, guarantees the convergence rate and the error probability, without any scalability issues. Therefore, the presented methods provide a computationally efficient tool to analyze the asynchronous numerical schemes.

VIII. CONCLUSIONS

This paper studied the stability, convergence rate, and error probability of the asynchronous parallel numerical algorithm. The asynchronous algorithm achieves better performance in terms of the total simulation time, particularly when massively parallel computing is required because it doesn't wait for synchronization across PEs. In order to analyze the asynchronous numerical algorithm, we adopted the switched linear system framework. Although modeling of massively parallel numerical algorithms as switched dynamical systems results in a very large number of modes, we developed new methods that circumvent this scalability issue. While the results presented here are based on 1D heat equation, the analysis approach is generic and be applicable to other PDEs as well.

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